Computer Modeling

Ordinary differential equations
continued

Homogeneous and nonhomogeneous systems

$$
\begin{aligned}
y_1'(t) &= a_{11}(t)y_1(t) + \cdots + a_{1M}(t)y_M(t) + u_1(t) \\
y_2'(t) &= a_{21}(t)y_1(t) + \cdots + a_{2M}(t)y_M(t) + u_2(t) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
y_M'(t) &= a_{1M}(t)y_1(t) + \cdots + a_{MM}(t)y_M(t) + u_M(t)
\end{aligned}
$$

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\begin{aligned}
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y_2'(t) &= a_{21}(t)y_1(t) + \cdots + a_{2M}(t)y_M(t) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (*) \\
y_M'(t) &= a_{1M}(t)y_1(t) + \cdots + a_{MM}(t)y_M(t)
\end{aligned}
$$

Homogeneous and nonhomogeneous systems

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&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
y_M'(t) &= a_{1M}(t)y_1(t) + \cdots + a_{MM}(t)y_M(t) + u_M(t)
\end{aligned}
$$

$$
\boxed{
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&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (*) \\
y_M'(t) &= a_{1M}(t)y_1(t) + \cdots + a_{MM}(t)y_M(t)
\end{aligned}
}
$$

General solutions

M linearly independent solutions of a homogeneous system of equations

$y_{11}(t), \cdots\cdots\cdots, y_{M1}(t)$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$ (**)

$y_{1M}(t), \cdots\cdots\cdots, y_{MM}(t)$

are called a fundamental system of solutions.

$$\begin{bmatrix} y_{11}(t) \\ \vdots \\ y_{M1}(t) \end{bmatrix} \cdots \begin{bmatrix} y_{1M}(t) \\ \cdots \\ y_{MM}(t) \end{bmatrix}$$

$$W(t) = \begin{vmatrix} y_{11}(t) & \cdots & y_{1M}(t) \\ \vdots & \ddots & \vdots \\ y_{M1}(t) & \cdots & y_{MM}(t) \end{vmatrix} \neq 0$$

for any $t \in (b, d)$.

$W(t)$ – the Wronskian (determinant)

normed fundamental matrix

(state transition matrix)

$\mathbf{\Phi}(t, t_0) = \mathbf{Y}(t)\mathbf{Y}^{-1}(t_0)$

$\boxed{\mathbf{\Phi}(t_0, t_0) = \mathbf{I}}$ – identity matrix

$\mathbf{\Phi}(t, t_0)$ is a solution of (&) with the initial condition $\mathbf{Y}(t_0) = \mathbf{I}$

Matrix equation

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_M \end{bmatrix} \qquad \boldsymbol{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_M \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \cdots \\ a_{M1} & \cdots & a_{MM} \end{bmatrix}$$

$$\frac{d\boldsymbol{y}(t)}{dt} = \mathbf{A}(t)\boldsymbol{y}(t) + \boldsymbol{u}(t)$$

$$\frac{d\boldsymbol{y}(t)}{dt} = \begin{bmatrix} \frac{dy_1(t)}{dt} \\ \vdots \\ \frac{dy_M(t)}{dt} \end{bmatrix}$$

$$y_1'(t) = a_{11}(t)y_1(t) + a_{12}(t)y_2(t)$$
$$y_2'(t) = a_{21}(t)y_1(t) + a_{22}(t)y_2(t)$$
$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

a homogeneous matrix equation

$$\frac{d\boldsymbol{y}(t)}{dt} = \mathbf{A}(t)\boldsymbol{y}(t)$$

a general solution

$$\boldsymbol{y}(t) = \mathbf{Y}(t)\boldsymbol{h}$$

a fundamental (solution) matrix
[nonsingular, Wronskian positive]

$$\mathbf{Y}(t) = \begin{bmatrix} y_{11}(t) & \cdots & y_{1M}(t) \\ \vdots & \ddots & \cdots \\ y_{M1}(t) & \cdots & y_{MM}(t) \end{bmatrix}$$

$$\boldsymbol{h} = \begin{bmatrix} h_1 \\ \vdots \\ h_M \end{bmatrix}$$

Nonhomogeneous equations

$\frac{d\boldsymbol{y}(t)}{dt} = \mathbf{A}(t)\boldsymbol{y}(t) + \boldsymbol{u}(t)$ ($)

Let's try a solution

$\boldsymbol{y}(t) = \boldsymbol{y}^s(t) + \boldsymbol{y}^0(t)$

$\boldsymbol{y}^s(t)$ – a particular solution of ($)

$\boldsymbol{y}^0(t) = \mathbf{Y}(t)\boldsymbol{h}$ – a general solution of the homogeneous equation

$$\frac{d\boldsymbol{y}^0(t)}{dt} = \mathbf{A}(t)\boldsymbol{y}^0(t)$$

$$\frac{d\left(\boldsymbol{y}^s(t)+\boldsymbol{y}^0(t)\right)}{dt} = \mathbf{A}(t)\boldsymbol{y}^s(t) +$$

$$+\mathbf{A}(t)\boldsymbol{y}^0(t) + \boldsymbol{u}(t)$$

A general solution of a nonhomogenous equation = a particular solution of the nonhomogenous equation + a general solution of the homogenous equation

Variation of parameters method

$\boldsymbol{y}^s(t) = \mathbf{Y}(t)\boldsymbol{h}(t)$ $\mathbf{Y}(t)$ - a fundamental matrix

$$\cancel{\frac{d\mathbf{Y}(t)}{dt}\boldsymbol{h}(t)} + \mathbf{Y}(t)\frac{d\boldsymbol{h}(t)}{dt} = \cancel{\mathbf{A}(t)\mathbf{Y}(t)\boldsymbol{h}(t)} + \boldsymbol{u}(t)$$

$$\mathbf{Y}(t)\frac{d\boldsymbol{h}(t)}{dt} = \boldsymbol{u}(t)$$

$$\frac{d\boldsymbol{h}(t)}{dt} = \mathbf{Y}^{-1}(\tau)\boldsymbol{u}(\tau)$$

$$\boldsymbol{h}(t) = \int_{t_0}^{t} \mathbf{Y}^{-1}(\tau)\boldsymbol{u}(\tau)d\tau \qquad \boldsymbol{h}(t_0) = \mathbf{0}$$

$$\boldsymbol{y}^s(t) = \mathbf{Y}(t)\int_{t_0}^{t} \mathbf{Y}^{-1}(\tau)\boldsymbol{u}(\tau)d\tau$$

$$\boxed{\boldsymbol{y}(t) = \mathbf{Y}(t)\boldsymbol{h} + \mathbf{Y}(t)\int_{t_0}^{t} \mathbf{Y}^{-1}(\tau)\boldsymbol{u}(\tau)d\tau}$$

General solution of the nonhomogenous equation

General solution of nonhomogeneous equations

$$\boldsymbol{y}^s(t) = \mathbf{Y}(t)\int_{t_0}^{t}\mathbf{Y}^{-1}(\tau)\boldsymbol{u}(\tau)d\tau$$

$$\boldsymbol{y}(t) = \mathbf{Y}(t)\boldsymbol{h} + \mathbf{Y}(t)\int_{t_0}^{t}\mathbf{Y}^{-1}(\tau)\boldsymbol{u}(\tau)d\tau$$

$$\boldsymbol{y}(t) = \mathbf{Y}(t)\mathbf{Y}^{-1}(t_0)\mathbf{Y}(t_0)\boldsymbol{h} +$$

$$+\mathbf{Y}(t)\mathbf{Y}^{-1}(t_0)\int_{t_0}^{t}\left(\mathbf{Y}(\tau)\mathbf{Y}^{-1}(t_0)\right)^{-1}\boldsymbol{u}(\tau)d\tau =$$

$$= \boldsymbol{\Phi}(t,t_0)\boldsymbol{b} + \boldsymbol{\Phi}(t,t_0)\int_{t_0}^{t}\boldsymbol{\Phi}^{-1}(\tau,t_0)\boldsymbol{u}(\tau)d\tau$$

$$\boxed{\boldsymbol{y}(t) = \boldsymbol{\Phi}(t,t_0)\boldsymbol{b} + \int_{t_0}^{t}\boldsymbol{\Phi}(t,\tau)\boldsymbol{u}(\tau)d\tau}$$

$\boldsymbol{\Phi}(t,\tau)$ or $\mathbf{Y}(t)$ has to be known!

$$\boldsymbol{b} = \mathbf{Y}(t_0)\boldsymbol{h}$$

$$\frac{dy(t)}{dt} = a(t)y(t) \qquad y(t_0) = y_0$$

$$\frac{dy}{y} = a(t) \quad y(t) = y_0\, e^{\int_{t_0}^{t} a(\tau)d\tau}$$

A question

$$\frac{d\boldsymbol{y}(t)}{dt} = \mathbf{A}(t)\boldsymbol{y}(t)$$

Is it true that $\boldsymbol{y}(t) = e^{\int_{t_0}^{t}\mathbf{A}(\tau)d\tau}\boldsymbol{y}_0$?

$$\int \mathbf{A}dt = \begin{bmatrix} \int a_{11}(t)dt & \cdots & \int a_{1M}(t)dt \\ \vdots & \ddots & \vdots \\ \int a_{M1}(t)dt & \cdots & \int a_{MM}(t)dt \end{bmatrix}$$

true only if the matrix and its integral commute

$$\mathbf{A}(\mathrm{t})\int \mathbf{A}(\tau)d\tau = \int \mathbf{A}(\tau)d\tau\ \mathbf{A}(t)$$

Function $e^{\mathbf{A}t}$

$$e^a = \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n$$

$$e^a = \sum_{i=0}^{\infty}\frac{a^i}{i!}$$

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{(\mathbf{A}t)^2}{2!} + \cdots = \sum_{i=0}^{\infty}\frac{\mathbf{A}^i t^i}{i!}$$

M^2 power series – convergent

$\mathbf{A}(t)$ continuous on $b \le t \le d$, then the series converges uniformly

$$\mathbf{A} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \mathbf{0} \qquad e^{\mathbf{A}t} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \mathbf{I} \qquad e^{\mathbf{A}t} = e^t\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$\frac{de^{\mathbf{A}t}}{dt} = \mathbf{A} + \mathbf{A}^2 t + \mathbf{A}^3\frac{t^2}{2!} + \cdots = \mathbf{A}e^{\mathbf{A}t}$$

1 $\mathbf{S}$ nonsingular

$$e^{\mathbf{S}\mathbf{A}\mathbf{S}^{-1}} = \mathbf{S}e^{\mathbf{A}}\mathbf{S}^{-1} \quad \text{(from the definition)}$$

$\boxed{\mathbf{I} = \mathbf{S}\mathbf{I}\mathbf{S}^{-1}}$ $\boxed{(\mathbf{S}\mathbf{A}\mathbf{S}^{-1}\mathbf{S}\mathbf{A}\mathbf{S}^{-1} = \mathbf{S}\mathbf{A}^2\mathbf{S}^{-1})}$

(proof from the definition)

$$\boxed{e^{\mathbf{A}_1}e^{\mathbf{A}_2} = e^{\mathbf{A}_2}e^{\mathbf{A}_1} \quad \text{if } \mathbf{A}_1, \mathbf{A}_2 \text{ commute}}$$

2 $\mathbf{A}_1, \mathbf{A}_2$ commute $(\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_2\mathbf{A}_1)$

$$e^{\mathbf{A}_1+\mathbf{A}_2} = e^{\mathbf{A}_1}e^{\mathbf{A}_2}$$

corrolaries

$$e^{\mathbf{A}(t_1+t_2)} = e^{\mathbf{A}t_1}e^{\mathbf{A}t_2} \qquad (\mathbf{A}t_1, \mathbf{A}t_2 \text{ commute})$$

$$\left(e^{\mathbf{A}}\right)^{-1} = e^{-\mathbf{A}} \qquad (t_1 = 1,\ t_2 = -1)$$

$$3\ \mathbf{A} = \begin{bmatrix}\mathbf{A}_1 & & 0\\ & \ddots & \\ 0 & & \mathbf{A}_m\end{bmatrix} \qquad e^{\mathbf{A}} = \begin{bmatrix}e^{\mathbf{A}_1} & & 0\\ & \ddots & \\ 0 & & e^{\mathbf{A}_m}\end{bmatrix}$$

because

$$\mathbf{A}^i = \begin{bmatrix}\mathbf{A}_1^i & & 0\\ & \ddots & \\ 0 & & \mathbf{A}_m^i\end{bmatrix}$$

4 $\mathbf{A}(t), \frac{d\mathbf{A}(t)}{dt}$ commute $\left(\mathbf{A}(t)\frac{d\mathbf{A}(t)}{dt} = \frac{d\mathbf{A}(t)}{dt}\mathbf{A}(t)\right)$

$$\frac{de^{\mathbf{A}(t)}}{dt} = \frac{d\mathbf{A}(t)}{dt}e^{\mathbf{A}(t)}$$

$$\boxed{\begin{matrix}\det e^{\mathbf{A}} = e^{\mathrm{tr}\mathbf{A}} > 0\\ \mathrm{tr}\,\mathbf{A} = a_{11} + \cdots + a_{MM}\end{matrix}}$$

A fundamental matrix

$$e^{\int_{t_0}^{t} \mathbf{A}(\tau)d\tau} \ ?$$

$$\frac{d\boldsymbol{Y}(t)}{dt} = \mathbf{A}(t)\boldsymbol{Y}(t) \qquad \boldsymbol{Y}(0) = \mathbf{I}$$

Is it true that $\boldsymbol{Y}(t) = e^{\int_{t_0}^{t} \mathbf{A}(\tau)d\tau}$?

$$e^{\int_{t_0}^{t} \mathbf{A}(\tau)d\tau} = \mathbf{I} + \int_{t_0}^{t} \mathbf{A}(\tau)d\tau +$$
$$+\frac{1}{2!}\left(\int_{t_0}^{t} \mathbf{A}(\tau)d\tau\right)^2 + \frac{1}{3!}\left(\int_{t_0}^{t} \mathbf{A}(\tau)d\tau\right)^3 \cdots$$

$$\frac{d}{dt}e^{\int_{t_0}^{t} \mathbf{A}(\tau)d\tau} = \mathbf{A}(t) + \frac{1}{2!}\mathbf{A}(t)\int_{t_0}^{t} \mathbf{A}(\tau)d\tau +$$
$$+\frac{1}{2!}\int_{t_0}^{t} \mathbf{A}(\tau)d\tau\,\mathbf{A}(t) + \frac{1}{3!}\mathbf{A}(t)\left(\int_{t_0}^{t} \mathbf{A}(\tau)d\tau\right)^2 +$$
$$+\frac{1}{3!}\int_{t_0}^{t} \mathbf{A}(\tau)d\tau\,\mathbf{A}(t)\int_{t_0}^{t} \mathbf{A}(\tau)d\tau +$$
$$+\frac{1}{3!}\left(\int_{t_0}^{t} \mathbf{A}(\tau)d\tau\right)^2 \mathbf{A}(t) + \cdots =$$

If $\mathbf{A}(t)\int_{t_0}^{t} \mathbf{A}(\tau)d\tau = \int_{t_0}^{t} \mathbf{A}(\tau)d\tau\,\mathbf{A}(t)$
(commute)

$$\frac{d}{dt}e^{\int_{t_0}^{t} \mathbf{A}(\tau)d\tau} = \mathbf{A}(t)\Big[\mathbf{I} + \int_{t_0}^{t} \mathbf{A}(\tau)d\tau +$$
$$\frac{1}{2!}\left(\int_{t_0}^{t} \mathbf{A}(\tau)d\tau\right)^2 + \frac{1}{3!}\left(\int_{t_0}^{t} \mathbf{A}(\tau)d\tau\right)^3 \cdots\Big] =$$
$$= \mathbf{A}(t)e^{\int_{t_0}^{t} \mathbf{A}(\tau)d\tau}$$

In this case, $\boldsymbol{Y}(t) = e^{\int_{t_0}^{t} \mathbf{A}(\tau)d\tau}$ (nonsingular) is a fundamental matrix, and it is the fundamental matrix normed at t_0, i.e.

$$\boldsymbol{\Phi}(t, t_0) = e^{\int_{t_0}^{t} \mathbf{A}(\tau)d\tau}$$

<u>Example</u> $\mathbf{A}(t)$=a(t)**B** **B** – a constant matrix

Equations with constant coefficients

commute

$\frac{d\boldsymbol{y}(t)}{dt} = \mathbf{A}\boldsymbol{y}(t) \qquad \boldsymbol{y}(t_0) = \boldsymbol{y}_0$

$\boxed{\boldsymbol{y}(t) = e^{\mathbf{A}(t-t_0)}\boldsymbol{y}_0} \quad \mathbf{A}\int \mathbf{A}d\tau = \int \mathbf{A}d\tau\ \mathbf{A}$

$e^{\mathbf{A}(t-t_0)}$ the normed fundamental matrix

for $t = t_0$ $(= \mathbf{I})$

any fundamental matrix $\mathbf{Y}(t)$ satisfies

$$\frac{d\mathbf{Y}(t)}{dt} = \mathbf{A}\mathbf{Y}(t)$$

$\frac{d\boldsymbol{Y}(t)}{dt} = \mathbf{A}\boldsymbol{Y}(t) \qquad \boldsymbol{Y}(0) = \mathbf{I}$

$s\mathcal{L}\{\boldsymbol{Y}(t)\} - \mathbf{I} = \mathbf{A}\mathcal{L}\{\boldsymbol{Y}(t)\} \quad \mathcal{L}\{\boldsymbol{Y}(t)\} = \boldsymbol{Y}(s)$

$(s\mathbf{I} - \mathbf{A})\mathcal{L}\{\boldsymbol{Y}(t)\} = \mathbf{I}$ [$(s\mathbf{I} - \mathbf{A})$ nonsingular generically]

$\mathcal{L}\{\boldsymbol{Y}(t)\} = (s\mathbf{I} - \mathbf{A})^{-1}$

$$\boxed{e^{\mathbf{A}t} = \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\}}$$

$$\frac{d\boldsymbol{y}(t)}{dt} = \mathbf{A}\boldsymbol{y}(t) + \boldsymbol{u}(t)$$

a general solution

$$\boldsymbol{y}(t) = e^{\mathbf{A}(t-t_0)}\boldsymbol{h} + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\boldsymbol{u}(\tau)d\tau$$

a particular solution for $\boldsymbol{y}(t_0) = \boldsymbol{y}_0$

$$\boxed{\boldsymbol{y}(t) = e^{\mathbf{A}(t-t_0)}\boldsymbol{y}_0 + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\boldsymbol{u}(\tau)d\tau}$$

$s\boldsymbol{y}(s) = \mathbf{A}\boldsymbol{y}(s) + \boldsymbol{u}(s) \qquad (\boldsymbol{y}(t_0) = \mathbf{0})$

$(s\mathbf{I} - \mathbf{A})\ \boldsymbol{y}(s) = \boldsymbol{u}(s)$

$$\boxed{\boldsymbol{y}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\boldsymbol{u}(s)}$$

$\mathbf{G}(s) = (s\mathbf{I} - \mathbf{A})^{-1}$ the transfer matrix

$\mathbf{G}(s) = \mathcal{L}\{\boldsymbol{Y}(t)\}$

Nonsingularity of $e^{\mathbf{A}}$

$\det e^{\mathbf{A}} = e^{\mathrm{tr}\mathbf{A}}$

$\mathrm{tr}\,\mathbf{A} = a_{11} + \cdots + a_{MM}$

<u>Sketch of a proof</u>

$\frac{d\mathbf{Z}(t)}{dt} = \mathbf{AZ}(t), \qquad \mathbf{Z}(0) = \mathbf{I}$

Solution: $\mathbf{Z}(t) = e^{\mathbf{A}t}$ a fundamental matrix

$M = 2$

$$\begin{bmatrix} \frac{dz_{11}}{dt} & \frac{dz_{12}}{dt} \\ \frac{dz_{21}}{dt} & \frac{dz_{22}}{dt} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{11}z_{11} + a_{12}z_{21} & a_{11}z_{12} + a_{12}z_{22} \\ a_{21}z_{11} + a_{22}z_{21} & a_{21}z_{12} + a_{22}z_{22} \end{bmatrix}$$

$\det e^{\mathbf{A}t} = \det \mathbf{Z}(\boldsymbol{t})$

$$\frac{d}{dt}\begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix} = \frac{d}{dt}(z_{11}z_{22} - z_{12}z_{21}) =$$

$$= \frac{dz_{11}}{dt}z_{22} + z_{11}\frac{dz_{22}}{dt} - \frac{dz_{12}}{dt}z_{21} - z_{12}\frac{dz_{21}}{dt} =$$

$$= \begin{vmatrix} \frac{dz_{11}}{dt} & \frac{dz_{12}}{dt} \\ z_{21} & z_{22} \end{vmatrix} + \begin{vmatrix} z_{11} & z_{12} \\ \frac{dz_{21}}{dt} & \frac{dz_{22}}{dt} \end{vmatrix} =$$

$$= \begin{vmatrix} a_{11}z_{11} + a_{12}z_{21} & a_{11}z_{12} + a_{12}z_{22} \\ z_{21} & z_{22} \end{vmatrix} +$$

$$+ \begin{vmatrix} z_{11} & z_{12} \\ a_{21}z_{11} + a_{22}z_{21} & a_{21}z_{12} + a_{22}z_{22} \end{vmatrix} =$$

$$= \begin{vmatrix} a_{11}z_{11} & a_{11}z_{12} \\ z_{21} & z_{22} \end{vmatrix} + \begin{vmatrix} z_{11} & z_{12} \\ a_{22}z_{21} & a_{22}z_{22} \end{vmatrix} =$$

$$= (a_{11} + a_{22}) \det \mathbf{Z} = \mathrm{tr}\,\mathbf{A} \det \mathbf{Z}$$

$\frac{d \det \mathbf{Z}(t)}{dt} = \mathrm{tr}\,\mathbf{A} \det \mathbf{Z}(t) \qquad \det \mathbf{Z}(0) = 1$

$\det \mathbf{Z}(t) = e^{\mathrm{tr}\mathbf{A}\,t} \qquad t = 1$

$\det e^{\mathbf{A}} = e^{\mathrm{tr}\mathbf{A}}$

Nonsingularity of $e^{\mathbf{A}}$ cont.

$$M = 3$$

$$\begin{bmatrix} \frac{dz_{11}}{dt} & \frac{dz_{12}}{dt} & \frac{dz_{13}}{dt} \\ \frac{dz_{21}}{dt} & \frac{dz_{22}}{dt} & \frac{dz_{23}}{dt} \\ \frac{dz_{31}}{dt} & \frac{dz_{32}}{dt} & \frac{dz_{33}}{dt} \end{bmatrix} =$$

$$= \begin{bmatrix} \sum_{i=1}^{3} a_{1i} z_{i1} & \sum_{i=1}^{3} a_{1i} z_{i2} & \sum_{i=1}^{3} a_{1i} z_{i3} \\ \sum_{i=1}^{3} a_{2i} z_{i1} & \sum_{i=1}^{3} a_{2i} z_{i2} & \sum_{i=1}^{3} a_{2i} z_{i3} \\ \sum_{i=1}^{3} a_{3i} z_{i1} & \sum_{i=1}^{3} a_{3i} z_{i2} & \sum_{i=1}^{3} a_{3i} z_{i3} \end{bmatrix}$$

Laplace expansion along the 1st row

$$\begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} = z_{11} \begin{vmatrix} z_{22} & z_{23} \\ z_{32} & z_{33} \end{vmatrix} -$$

$$-z_{12} \begin{vmatrix} z_{21} & z_{13} \\ z_{31} & z_{33} \end{vmatrix} + z_{13} \begin{vmatrix} z_{21} & z_{22} \\ z_{31} & z_{32} \end{vmatrix}$$

$$\frac{d}{dt} \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} = \frac{dz_{11}}{dt} \begin{vmatrix} z_{22} & z_{23} \\ z_{32} & z_{33} \end{vmatrix} +$$

$$z_{11} \frac{d}{dt} \begin{vmatrix} z_{22} & z_{23} \\ z_{32} & z_{33} \end{vmatrix} - \frac{dz_{12}}{dt} \begin{vmatrix} z_{21} & z_{23} \\ z_{31} & z_{33} \end{vmatrix} -$$

$$-z_{12} \frac{d}{dt} \begin{vmatrix} z_{21} & z_{13} \\ z_{31} & z_{33} \end{vmatrix} + \frac{dz_{13}}{dt} \begin{vmatrix} z_{21} & z_{22} \\ z_{31} & z_{32} \end{vmatrix} +$$

$$+z_{13} \frac{d}{dt} \begin{vmatrix} z_{21} & z_{22} \\ z_{31} & z_{32} \end{vmatrix} = \begin{vmatrix} \frac{dz_{11}}{dt} & \frac{dz_{12}}{dt} & \frac{dz_{13}}{dt} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} +$$

$$+ \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ \frac{dz_{21}}{dt} & \frac{dz_{22}}{dt} & \frac{dz_{23}}{dt} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} + \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ \frac{dz_{31}}{dt} & \frac{dz_{32}}{dt} & \frac{dz_{33}}{dt} \end{vmatrix} =$$

$$= \begin{vmatrix} a_{11} z_{11} & a_{11} z_{12} & a_{11} z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} + \cdots =$$

$$= (a_{11} + a_{22} + a_{33}) \det \mathbf{Z} = \operatorname{tr} \mathbf{A} \det \mathbf{Z}$$

and further as before

Jordan canonical form: distinct eigenvalues

$\frac{d\mathbf{Y}(t)}{dt} = \mathbf{A}\mathbf{Y}(t)$, transformation of $\mathbf{Y}(t)$ to

$\mathbf{Z}(t) = \mathbf{S}^{-1}\mathbf{Y}(t)$ $\quad$ $\mathbf{S}$ - a nonsingular matrix

$\frac{d\mathbf{Z}(t)}{dt} = \mathbf{\Lambda}\mathbf{Z}(t)$ $\quad$ $\mathbf{\Lambda} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$

$\mathbf{\Lambda}$ - Jordan canonical form $\quad$ $\mathbf{Z}(t) = e^{\mathbf{\Lambda}t}$

The form depends on eigenvalues of $\mathbf{A}$

$(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{x} = \mathbf{0}$ $\quad$ the characteristic equation

$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \rightarrow M$ eigenvalues λ_i

$\boldsymbol{x}_i$ - eigenvectors

<u>distinct (single) eigenvalues</u>:

columns of $\mathbf{S}$ are eigenvectors, as

$\mathbf{A}\mathbf{S} = \mathbf{S}\mathbf{\Lambda}$

$\mathbf{A}[\boldsymbol{s}_1, \dots, \boldsymbol{s}_M] = [\lambda_1\boldsymbol{s}_1, \dots, \lambda_M\boldsymbol{s}_M]$

$(\mathbf{A} - \lambda_1\mathbf{I})\boldsymbol{s}_1 = \mathbf{0}, \dots, (\mathbf{A} - \lambda_M\mathbf{I})\boldsymbol{s}_M = \mathbf{0}$

$$\boldsymbol{s}_i = \begin{bmatrix} s_{i1} \\ \vdots \\ s_{iM} \end{bmatrix} \neq \mathbf{0}$$

For distinct eigenvalues $\lambda_i \neq \lambda_j$ for $i \neq j$

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_M \end{bmatrix} \quad e^{\mathbf{\Lambda}t} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_M t} \end{bmatrix}$$

$$\mathbf{Y}(t) = \mathbf{S}\mathbf{Z}(t) = \begin{bmatrix} s_{11}e^{\lambda_1 t} & \cdots & s_{1M}e^{\lambda_M t} \\ \vdots & \ddots & \vdots \\ s_{M1}e^{\lambda_1 t} & \cdots & s_{MM}e^{\lambda_M t} \end{bmatrix}$$

$\mathbf{Y}(t)$ - a fundamental matrix (**Example (£)**)

a general solution

of the homogeneous equation

for distinct eigenvalues

$$\boxed{\boldsymbol{y}^0(t) = \mathbf{Y}(t)\boldsymbol{h} = \sum_{i=1}^{M} h_i\boldsymbol{s}_i e^{\lambda_i t} = \sum_{i=1}^{M} \boldsymbol{c}_i e^{\lambda_i t}}$$

An example (1)

$$\frac{dy_1(t)}{dt} = 5y_1(t) + 4y_2(t)$$
$$\frac{dy_2(t)}{dt} = 4y_1(t) + 5y_2(t)$$

the characteristic equation

$$\begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 9 = 0$$

$\lambda_1 = 1, \quad \lambda_2 = 9$

an eigenvector for $\lambda_1 = 1$

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} s_{11} \\ s_{21} \end{bmatrix} = \mathbf{0},\ s_{11} = 1, s_{21} = -1$$

an eigenvector for $\lambda_2 = 9$

$$\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} s_{12} \\ s_{22} \end{bmatrix} = \mathbf{0},\ s_{12} = 1, s_{22} = 1$$

$$W(t) = \begin{vmatrix} e^t & e^{9t} \\ -e^t & e^{9t} \end{vmatrix} = 2e^{10t} > 0$$

a fundamental matrix

$$Y(t) = \begin{bmatrix} e^t & e^{9t} \\ -e^t & e^{9t} \end{bmatrix}$$

a general solution

$y_1(t) = h_1 e^t + h_2 e^{9t}$

$y_2(t) = -h_1 e^t + h_2 e^{9t}$

checking

L_1: $h_1 e^t + 9h_2 e^{9t}$

P_1: $5(h_1 e^t + h_2 e^{9t}) + 4(-h_1 e^t + h_2 e^{9t}) = L_1$

L_2: $-h_1 e^t + 9h_2 e^{9t}$

P_2: $4(h_1 e^t + h_2 e^{9t}) + 5(-h_1 e^t + h_2 e^{9t}) = L_2$

An example (2)

$$\frac{dy_1(t)}{dt} = y_1(t) - y_2(t)$$
$$\frac{dy_2(t)}{dt} = y_1(t) + y_2(t)$$

the characteristic equation

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 2 = 0$$

$\lambda_1 = 1 + i, \quad \lambda_2 = 1 - i$

an eigenvector for $\lambda_1 = 1 + i$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} s_{11} \\ s_{21} \end{bmatrix} = \mathbf{0}, \; s_{11} = i, s_{21} = 1$$

an eigenvector for $\lambda_2 = 1 - i$

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} s_{12} \\ s_{22} \end{bmatrix} = \mathbf{0}, \; s_{12} = i, s_{22} = -1$$

$$y_{11}(t) = ie^{(i+1)t} = -e^t \sin t + ie^t \cos t$$
$$y_{21}(t) = e^{(i+1)t} = e^t \cos t + ie^t \sin t$$
$$y_{12}(t) = ie^{(i-1)t} = e^t \sin t + ie^t \cos t$$
$$y_{22}(t) = -e^{(i-1)t} = -e^t \cos t + ie^t \sin t$$

$$W(t) = \begin{vmatrix} e^t \cos t & -e^t \sin t \\ e^t \sin t & e^t \cos t \end{vmatrix} = e^{2t} > 0$$

a fundamental matrix

$$Y(t) = \begin{bmatrix} e^t \cos t & -e^t \sin t \\ e^t \sin t & e^t \cos t \end{bmatrix}$$

a general solution

$$y_1(t) = h_1 e^t \cos t - h_2 e^t \sin t$$
$$y_2(t) = h_1 e^t \sin t + h_2 e^t \cos t$$

Jordan canonical form: multiple eigenvalues

$\mathbf{\Lambda} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{\Lambda}_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{\Lambda}_L \end{bmatrix}$$

L – number of different eigenvalues

$$\mathbf{\Lambda}_l = \begin{bmatrix} \mathbf{\Lambda}_{l1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{\Lambda}_{lk_l} \end{bmatrix}$$

k_l - multiplicity of the l-th eigenvalue

$$\mathbf{\Lambda}_{lj} = \begin{bmatrix} \lambda_l & 0 & 0 & \cdots & 0 & 0 \\ 1 & \lambda_l & 0 & \cdots & 0 & 0 \\ 0 & 1 & \lambda_l & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \lambda_l \end{bmatrix}$$

a Jordan block

$$\mathbf{\Lambda}_{lj} = \lambda_l \mathbf{I} + \mathbf{F}_{lj} \qquad \mathbf{F}_{lj} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$e^{\mathbf{\Lambda}t} = \begin{bmatrix} e^{\mathbf{\Lambda}_{11}t} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & e^{\mathbf{\Lambda}_{Lk_L}t} \end{bmatrix}$$

$$e^{\mathbf{\Lambda}_{lj}t} = e^{(\lambda_l \mathbf{I} + \mathbf{F}_{lj})t} = e^{\lambda_l t} e^{\mathbf{F}_{lj}t}$$

$\mathbf{I}$ and $\mathbf{F}_{lj}$ commute

$$e^{\mathbf{F}_{lj}t} = \mathbf{I} + \mathbf{F}_{lj}t + \frac{1}{2!}\mathbf{F}_{lj}^2 t^2 + \cdots$$

<u>assume</u> $k_l = 3$

$$\mathbf{F}_{lj} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \mathbf{F}_{lj}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \mathbf{F}_{lj}^3 = \mathbf{0}$$

$$e^{\mathbf{\Lambda}_{lj}t} = e^{\lambda_l t}\left(\mathbf{I} + \mathbf{F}_{lj}t + \frac{1}{2!}\mathbf{F}_{lj}^2 t^2\right) = e^{\lambda_l t} \begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ \frac{1}{2!}t^2 & t & 1 \end{bmatrix}$$

<u>in general</u>

$$\boxed{\boldsymbol{y}(t) = \sum_{l=1}^{L} \sum_{j=0}^{k_l - 1} t^j e^{\lambda_l t} \mathbf{Z}_{lj} \boldsymbol{h}}$$

$\mathbf{Z}_{lj}$ depends only on $\mathbf{A}$, not on t

Jordan canonical form: multiple eigenvalues

$$\boldsymbol{y}(t) = \sum_{l=1}^{L} \sum_{j=0}^{k_l-1} t^j e^{\lambda_l t} \mathbf{Z}_{lj} \boldsymbol{h} =$$

$$= e^{\lambda_1 t}\, \boldsymbol{c}_{10} + t e^{\lambda_1 t} \boldsymbol{c}_{11} + \cdots + t^{k_1-1} e^{\lambda_1 t} \boldsymbol{c}_{1,k_1-1} +$$

… … … … … … … … … … … … … … … … …

$$+ e^{\lambda_L t} \boldsymbol{c}_{L0} + t e^{\lambda_L t} \boldsymbol{c}_{L1} + \cdots + t^{k_L-1} e^{\lambda_L t} \boldsymbol{c}_{L,k_L-1}$$

Jordan decomposition is numericaly unstable

Example

$$\mathbf{A} = \begin{bmatrix} 1 & \varepsilon \\ 1 & 1 \end{bmatrix}$$

For $\varepsilon = 0$ the Jordan form

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

For $\varepsilon > 0$

$$\begin{vmatrix} 1-\lambda & \varepsilon \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 - \varepsilon = 0$$

$$\lambda_1 = 1 - \sqrt{\varepsilon}, \qquad \lambda_2 = 1 + \sqrt{\varepsilon}$$

$$\begin{bmatrix} \sqrt{\varepsilon} & \varepsilon \\ 1 & \sqrt{\varepsilon} \end{bmatrix} \begin{bmatrix} s_{11} \\ s_{21} \end{bmatrix} = 0 \quad \begin{matrix} s_{11} = -\sqrt{\varepsilon} \\ s_{21} = 1 \end{matrix}$$

$$\begin{bmatrix} -\sqrt{\varepsilon} & \varepsilon \\ 1 & -\sqrt{\varepsilon} \end{bmatrix} \begin{bmatrix} s_{12} \\ s_{22} \end{bmatrix} = 0 \quad \begin{matrix} s_{12} = 1 \\ s_{22} = \sqrt{\varepsilon} \end{matrix}$$

$$\mathbf{S} = \begin{bmatrix} -\sqrt{\varepsilon} & 1 \\ 1 & \sqrt{\varepsilon} \end{bmatrix} \qquad \mathbf{S}^{-1} = \begin{bmatrix} -\frac{1}{2\sqrt{\varepsilon}} & \frac{1}{2} \\ \frac{1}{2\sqrt{\varepsilon}} & \frac{1}{2} \end{bmatrix}$$

$$\mathbf{\Lambda} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} -\frac{1}{2\sqrt{\varepsilon}} & \frac{1}{2} \\ \frac{1}{2\sqrt{\varepsilon}} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & \varepsilon \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -\sqrt{\varepsilon} & 1 \\ 1 & \sqrt{\varepsilon} \end{bmatrix} =$$

$$= \begin{bmatrix} 1-\sqrt{\varepsilon} & 0 \\ 0 & 1+\sqrt{\varepsilon} \end{bmatrix}$$

Calculation of $e^{\mathbf{A}t}$

from the definition (1)

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \cdots = \sum_{i=0}^{\infty} \frac{\mathbf{A}^i t^i}{i!}$$

1. $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad e^{\mathbf{A}t} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2. $\mathbf{A} = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ $\quad e^{\mathbf{A}t} = \begin{bmatrix} 1 & bt \\ 0 & 1 \end{bmatrix}$

3. $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad e^{\mathbf{A}t} = e^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4. $\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ $\quad e^{\mathbf{A}t} = e^{at} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

5. $\mathbf{A} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ $\quad e^{\mathbf{A}t} = e^{at} \begin{bmatrix} 1 & bt \\ 0 & 1 \end{bmatrix}$

Hints:

2. $\mathbf{A}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

3. $\mathbf{A}^i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4. $\mathbf{A}^i = \begin{bmatrix} a^i & 0 \\ 0 & a^i \end{bmatrix}$

5. $\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ (commute)

$e^{\mathbf{A}_1 + \mathbf{A}_2} = e^{\mathbf{A}_1} e^{\mathbf{A}_2}$ $\quad (\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_2\mathbf{A}_1)$

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} 0 & ab \\ 0 & 0 \end{bmatrix}$$

Calculation of $e^{\mathbf{A}t}$ from the definition (2)

$$\boxed{e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \cdots = \sum_{i=0}^{\infty} \frac{\mathbf{A}^i t^i}{i!}}$$

6. $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad e^{\mathbf{A}t} = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}$

$$\mathbf{A}^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

$$\mathbf{A}^i = \begin{cases} \mathbf{A} & \text{for } i \text{ odd} \\ \mathbf{I} & \text{for } i \text{ even} \end{cases}$$

$$\sum_{i=0}^{\infty} \frac{\mathbf{A}^i t^i}{i!} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{I}t^2}{2!} + \frac{\mathbf{A}t^3}{2!} + \cdots =$$

$$= \begin{bmatrix} 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots & t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots \\ t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots & 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{2}(e^t + e^{-t}) & \frac{1}{2}(e^t - e^{-t}) \\ \frac{1}{2}(e^t - e^{-t}) & \frac{1}{2}(e^t + e^{-t}) \end{bmatrix} =$$

$$= \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}$$

$$\boxed{e^t + e^{-t} = 1 + t + \frac{t^2}{2!} + \cdots 1 - t + \frac{t^2}{2!} + \cdots}$$

7. $\mathbf{B} = \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} \quad e^{\mathbf{B}t} = \begin{bmatrix} \cosh bt & \sinh bt \\ \sinh bt & \cosh bt \end{bmatrix}$

$$\mathbf{B} = \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} = b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = b\mathbf{A}$$

$$\mathbf{B}^2 = b^2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = b^2\mathbf{I} \quad \mathbf{B}^3 = b^3\mathbf{A}$$

$$\mathbf{B}^i = \begin{cases} b^i\mathbf{A} & \text{for } i \text{ odd} \\ b^i\mathbf{I} & \text{for } i \text{ even} \end{cases}$$

$$\sum_{i=0}^{\infty} \frac{\mathbf{B}^i t^i}{i!} = \mathbf{I} + \mathbf{A}bt + \frac{\mathbf{I}(bt)^2}{2!} + \frac{\mathbf{A}(bt)^3}{2!} + \cdots =$$

$$= \begin{bmatrix} 1 + \frac{(bt)^2}{2!} + \frac{(bt)^4}{4!} + \cdots & bt + \frac{(bt)^3}{3!} + \frac{(bt)^5}{5!} + \cdots \\ bt + \frac{(bt)^3}{3!} + \frac{(bt)^5}{5!} + \cdots & 1 + \frac{(bt)^2}{2!} + \frac{(bt)^4}{4!} + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}(e^{bt} + e^{-bt}) & \frac{1}{2}(e^{bt} - e^{-bt}) \\ \frac{1}{2}(e^{bt} - e^{-bt}) & \frac{1}{2}(e^{bt} + e^{-bt}) \end{bmatrix} =$$

$$= \begin{bmatrix} \cosh bt & \sinh bt \\ \sinh bt & \cosh bt \end{bmatrix}$$

Calculation of $e^{\mathbf{A}t}$

from the Cayley-Hamilton theorem

$\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^M + \alpha_{M-1}\lambda^{M-1} + \cdots + \alpha_0 = 0$

Matrix $\mathbf{A}$ satisfies its characteristic equation

$\mathbf{A}^M + \alpha_{M-1}\mathbf{A}^{M-1} + \cdots + \alpha_0\mathbf{I} = \mathbf{0}$

Minimal polynomial

$\mathbf{A}^J + \kappa_{M-1}\mathbf{A}^{J-1} + \cdots + \kappa_0\mathbf{I} = \mathbf{0} \quad J \le M$

The minimal polynomial is unique,

λ_i are the roots of the minimal polynomial,

but possibly of lower multiplicity.

Consider differential equation

$$\frac{d^J f(t)}{dt^J} + \kappa_{M-1}\frac{d^{J-1} f(t)}{dt^{J-1}} + \cdots + \kappa_0 f(t) = 0$$

Let a set of functions

$f_p(t), \quad p = 0,1,\ldots,J-1$

be the particular solutions satisfying the initial conditions

$$f_p^{(r)}(0) = \begin{cases} 0 & \text{for} \quad r \neq p \\ 1 & \text{for} \quad r = p \end{cases}$$

Then

$$\boxed{e^{\mathbf{A}t} = f_{J-1}(t)\mathbf{A}^{J-1} + \cdots + f_0(t)\mathbf{I}}$$

An example

$$\mathbf{A} = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad \det(\mathbf{A} - \lambda\mathbf{I}) =$$

$$= \begin{vmatrix} 1-\lambda & 5 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^3$$

$\lambda = 1$ – the triple eigenvalue

$(\mathbf{I} - \mathbf{A})^3 = \mathbf{0}$ from the Cayley-Hamilton theorem

$$\begin{bmatrix} 0 & -5 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -5 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The minimal polynomial ($J = 2$)

$(\lambda - 1)^2 = \lambda^2 - 2\lambda + 1 = 0$

$$\frac{d^2 f(t)}{dt^2} - 2\frac{df(t)}{dt} + f(t) = 0$$

solutions: $f_1(t) = e^t \quad f_2(t) = te^t$

$g_1(t) = f(t),\ \ g_2(t) = f'(t),\ \ \frac{d}{dt}\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$

$$W(t) = \begin{vmatrix} f_{11}(t) & f_{21}(t) \\ f'_{11}(t) & f'_{22}(t) \end{vmatrix} = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} > 0$$

a general solution $f(t) = \gamma_1 e^t + \gamma_2 te^t$

the particular solutions

$f_0(0) = 1,\ f_0^{(1)}(0) = 0\colon\ f_0(t) = e^t - te^t$

$f_1(0) = 0,\ f_1^{(1)}(0) = 1\colon\ f_1(t) = te^t$

$$e^{\mathbf{A}t} = te^t \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} + (e^t - te^t)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & 5te^t & 0 \\ 0 & e^t & 0 \\ 0 & te^t & e^t \end{bmatrix}$$

$$\boxed{e^{\mathbf{A}t} = f_{J-1}(t)\mathbf{A}^{J-1} + \cdots + f_0(t)\mathbf{I}}$$

Calculation of $e^{\mathbf{A}t}$

from the Laplace transform

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\}$$

[$s \neq s_i$ - any eigenvalue]

<u>Lemma</u>

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\mathbf{B}(s)}{d(s)}$$

where

$$d(s) = \det(s\mathbf{I} - \mathbf{A}) = s^M + d_1 s^{M-1} + \cdots + d_{M-1}s + d_M$$

$$\mathbf{B}(s) = s^{M-1}\mathbf{B}_0 + s^{M-2}\mathbf{B}_1 + \cdots + s\mathbf{B}_{M-2} + \mathbf{B}_{M-1}$$

<u>Proof</u>: From the Cramer's rule, the (i,j) element of $(s\mathbf{I} - \mathbf{A})^{-1}$ is equal to $P_{ij}(s)/d(s)$, where $P_{ij}(s)$ is the cofactor of the (i,j) element. The cofactor is equal to $(-1)^{i+j}$ times the determinant of a $(M-1) \times (M-1)$ matrix.

<u>Formulae for matrices $\mathbf{B}_i$</u>

$$\mathbf{B}_0 = \mathbf{I} \qquad d_1 = -\mathrm{tr}\,\mathbf{A}$$

$$\mathbf{B}_1 = \mathbf{B}_0\mathbf{A} + d_1\mathbf{I} \qquad d_2 = -\frac{1}{2}\mathrm{tr}\,(\mathbf{B}_1\mathbf{A})$$

……………………… ……………………………

$$\mathbf{B}_k = \mathbf{B}_{k-1}\mathbf{A} + d_k\mathbf{I} \qquad d_k = -\frac{1}{k}\mathrm{tr}\,(\mathbf{B}_{k-1}\mathbf{A})$$

……………………… ……………………………

$$\mathbf{B}_{M-1} = \mathbf{B}_{M-2}\mathbf{A} + d_M\mathbf{I} \qquad d_{M-1} = -\frac{1}{M-1}\mathrm{tr}\,(\mathbf{B}_{M-2}\mathbf{A})$$

$$d_M = -\frac{1}{M}\mathrm{tr}\,(\mathbf{B}_{M-1}\mathbf{A})$$

<u>Comment</u>

The elements of the $(s\mathbf{I} - \mathbf{A})^{-1}$ matrix are rational functions of the variable s. It may happen that some or even all elemements have common factors both in the nominator and the denominator.

<u>Proof</u> L.A.Zadeh, C.A.Desoer (1963) *Linear System Theory*. McGraw-Hill, New York.

An example

$$\frac{dy_1(t)}{dt} = -2y_1(t) + y_2(t)$$
$$\frac{dy_2(t)}{dt} = 2y_1(t) - 3y_2(t)$$

$$\frac{d\boldsymbol{y}(t)}{dt} = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix} \boldsymbol{y}(t)$$

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s+2 & -1 \\ -2 & s+3 \end{bmatrix}$$

$$\begin{vmatrix} s+2 & -1 \\ -2 & s+3 \end{vmatrix} = s^2 + 5s + 4 = (s+1)(s+4)$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \frac{s+3}{(s+1)(s+4)} & \frac{1}{(s+1)(s+4)} \\ \frac{2}{(s+1)(s+4)} & \frac{s+2}{(s+1)(s+4)} \end{bmatrix}$$

$$\frac{s+3}{(s+1)(s+4)} = \frac{2}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s+4}$$
$$\frac{2}{(s+1)(s+4)} = \frac{2}{3}\frac{1}{s+1} - \frac{2}{3}\frac{1}{s+4}$$
$$\frac{1}{(s+1)(s+4)} = \frac{1}{3}\frac{1}{s+1} - \frac{1}{3}\frac{1}{s+4}$$
$$\frac{s+2}{(s+1)(s+4)} = \frac{1}{3}\frac{1}{s+1} + \frac{2}{3}\frac{1}{s+4}$$

$$e^{\mathbf{A}t} = \mathcal{L}^{-\mathbf{1}}\{(s\mathbf{I} - \mathbf{A})^{-1}\} =$$
$$= \frac{1}{3}\begin{bmatrix} 2e^{-t} + e^{-4t} & e^{-t} + 2e^{-4t} \\ 2e^{-t} - 2e^{-4t} & e^{-t} - e^{-4t} \end{bmatrix}$$

Equations with periodic coefficients

$\frac{d\mathbf{y}(t)}{dt} = \mathbf{A}(t)\mathbf{y}(t) \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0$

$\mathbf{A}(t)$ –piecewise continuous periodic function of t with a period T $\quad M \times M$

$\mathbf{A}(t+T) = \mathbf{A}(t)$

The solution

$\boldsymbol{y}(t) = \boldsymbol{\Phi}(t,0)\boldsymbol{y}_0 \stackrel{\text{def}}{=} \boldsymbol{\Phi}(t)\boldsymbol{y}_0$

$\boldsymbol{\Phi}(t)$ - the normed fundamental matrix

$\frac{d\boldsymbol{\Phi}(t)}{dt} = \mathbf{A}(t)\boldsymbol{\Phi}(t) \qquad \boldsymbol{\Phi}(0) = \mathbf{I}$

$\frac{d\boldsymbol{\Phi}(t+T)}{dt} = \mathbf{A}(t+T)\boldsymbol{\Phi}(t+T) =$

$= \mathbf{A}(t)\boldsymbol{\Phi}(t+T)$

$\boldsymbol{\Phi}(t+T)$ - a fundamental matrix

$\boldsymbol{\Phi}(T)$ $(t = 0)$ nonsingular

$\boldsymbol{\Psi}(t) \stackrel{\text{def}}{=} \boldsymbol{\Phi}(t+T)\boldsymbol{\Phi}^{-1}(T)$

$\frac{d\boldsymbol{\Psi}(t)}{dt} = \mathbf{A}(t)\boldsymbol{\Psi}(t) \qquad \boldsymbol{\Psi}(0) = \mathbf{I}$

$\boldsymbol{\Psi}(t)$ - the normed fundamental matrix

$\boldsymbol{\Psi}(t) = \boldsymbol{\Phi}(t)$

$\boldsymbol{\Phi}(t+T) = \boldsymbol{\Phi}(t)\,\boldsymbol{\Phi}(T)$

$\boldsymbol{\Phi}(t+nT) = \boldsymbol{\Phi}(t)\,\boldsymbol{\Phi}^n(T)$

$\boldsymbol{y}(t+nT) = \boldsymbol{\Phi}(t)\boldsymbol{\Phi}^n(T)\boldsymbol{y}_0$

$\boldsymbol{y}(nT) = \boldsymbol{\Phi}^n(T)\boldsymbol{y}_0$

Equations with periodic coefficients

consider an equation

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{B}\mathbf{x}(t) \qquad \boldsymbol{x}(0) = \boldsymbol{y}_0$$

$\mathbf{B}$ – a constant matrix

$$\mathbf{x}(t) = e^{\mathbf{B}t}\boldsymbol{y}_0$$

require $\quad \boldsymbol{y}(nT) = \mathbf{x}(nT)$

$$\boldsymbol{\Phi}^n(T)\boldsymbol{y}_0 = \left(e^{\mathbf{B}T}\right)^n \boldsymbol{y}_0$$

$\mathbf{C} = [\boldsymbol{y}_0^1, \boldsymbol{y}_0^2, \cdots, \boldsymbol{y}_0^M]$ - chosen to be nonsingular

$$\boldsymbol{\Phi}^n(T)\mathbf{C} = \left(e^{\mathbf{B}T}\right)^n \mathbf{C} \qquad \times \mathbf{C}^{-1}$$

$n = 1$

$$\boldsymbol{\Phi}(T) = e^{\mathbf{B}T}$$

$$\boldsymbol{\Phi}(t) = \boldsymbol{\Phi}(t)e^{-\mathbf{B}t}e^{\mathbf{B}t} \stackrel{\text{def}}{=} \mathbf{P}(t)e^{\mathbf{B}t}$$

A fundamental matrix in the Floquet normal form

$$\boxed{\boldsymbol{y}(t) = \mathbf{P}(t)e^{\mathbf{B}t}\boldsymbol{y}_0} \quad \mathbf{P}(0) = \boldsymbol{\Phi}(0) = \mathbf{I}$$

The Floquet solution

$\mathbf{P}(t)$ periodic, as

$$\mathbf{P}(t+T) = \boldsymbol{\Phi}(t+T)e^{-\mathbf{B}(t+T)} =$$

$$\boldsymbol{\Phi}(t)\,\boldsymbol{\Phi}(T)e^{-\mathbf{B}t}e^{-\mathbf{B}T} =$$

(exponents commute)

$$\boldsymbol{\Phi}(t)e^{\mathbf{B}T}e^{-\mathbf{B}T}e^{-\mathbf{B}t} =$$

$$= \boldsymbol{\Phi}(t)e^{-\mathbf{B}t} = \mathbf{P}(t)$$

$\mathbf{P}(t)$ continuous and periodic, thus bounded

Thank you!